Affine structures on abelian Lie Groups

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The Nagano-Yagi-Goldmann theorem states that on the torus $\mathbb{T}^2$, every affine (or projective) structure is invariant or is constructed on the basis of some Goldmann rings [12]. It shows the interest to study the invariant affine structure on the torus $\mathbb{T}^2$ or on abelian Lie groups. Recently, the works of Kim [8] and Dekimpe-Ongenae [5] precise the number of non equivalent invariant affine structures on an abelian Lie group in the case these structures are complete. In this paper we propose a study of complete and non complete affine structure on abelian Lie groups based on the geometry of the algebraic variety of finite dimensional associative algebras.

1 Affine structure on Lie groups and Lie algebras

The Lie group $Aff(\mathbb{R}^n)$ is the group of affine transformations of $\mathbb{R}^n$. It is constituted of matrices

$$\left(\begin{array}{cc} A & b \\ 0 & 1 \end{array} \right)$$

with $A \in GL(n, \mathbb{R})$, $b \in \mathbb{R}^n$. It acts on the real affine space $\widetilde{\mathbb{R}}^n$ by

$$\left(\begin{array}{cc} A & b \\ 0 & 1 \end{array} \right) \left(\begin{array}{c} v \\ 1 \end{array} \right) = \left(\begin{array}{c} Av + b \\ 1 \end{array} \right)$$

where $(v, 1)^t \in \widetilde{\mathbb{R}}^{n+1}$.

Its Lie algebra, noted $aff(\mathbb{R}^n)$, is the linear algebra

$$aff(\mathbb{R}^n) = \left\{ \left(\begin{array}{cc} A & b \\ 0 & 0 \end{array} \right) ; A \in gl(n, \mathbb{R}), b \in \mathbb{R}^n \right\}$$

Definition 1.1 *An affine structure on a Lie algebra $\mathfrak{g}$ is a morphism*

$$\Psi : \mathfrak{g} \to aff(\mathbb{R}^n)$$

of Lie algebras.

Remark. Let us consider an affine representation of a Lie group G that is an homomorphism $\varphi : G \to Aff(\mathbb{R}^n)$. For every $g \in G$, $\varphi(g)$ is an affine transformation on the affine space $\widetilde{\mathbb{R}}^{n+1}$. This representation induces an affine structure on the Lie algebra $\mathfrak{g}$ of G.

Proposition 1.1 *The Lie algebra $\mathfrak{g}$ is provided with an affine structure if and only if the underlying vector space $A(\mathfrak{g})$ is a left symmetric algebra, that is there exists a bilinear mapping*

$$
\begin{array}{ccc}
A(\mathfrak{g}) \times A(\mathfrak{g}) & \to & A(\mathfrak{g}) \\
(X, Y) & \mapsto & X.Y
\end{array}
$$

satisfying
1) $X.(Y.Z) - Y.(X.Z) = (X.Y).Z - (Y.X).Z$
2) $X.Y - Y.X = [X, Y]$
for every $X, Y, Z \in A(\mathfrak{g})$

If Ψ is a morphism giving an affine structure on $\mathfrak{g}$, then the left symmetric product on $A(\mathfrak{g})$ is defined as this:

$$
\forall X \in \mathfrak{g}, \Psi(X) = \left(\begin{array}{cc} A(X) & b(X) \\ 0 & 0 \end{array} \right)
$$

and we put
$$
X.Y = b^{-1}(A(X).b(X))
$$

where $b : A(\mathfrak{g}) \to \mathbb{R}^n$ is supposed to be bijective.

The fact that Ψ is a representation implies that $X.Y$ is a left symmetric product. Conversely, if $X.Y$ is a left symmetric product on $A(\mathfrak{g})$, and if L_X indicates the left representation : $L_X(Y) = X.Y$, then the map

$$
X \to \left(\begin{array}{cc} L_X & X \\ 0 & 0 \end{array} \right)
$$

defines an affine structure on the Lie algebra $\mathfrak{g}$.

If Ψ is an affine structure on $\mathfrak{g}$, it defines a representation

$$
\Psi(X) = \left(\begin{array}{cc} A(X) & b(X) \\ 0 & 0 \end{array} \right)
$$

and a left symmetric product $X.Y$. This last induces an affine representation

$$
\left(\begin{array}{cc} L_X & X \\ 0 & 0 \end{array} \right)
$$

which is equivalent to Ψ (and equal if $b = Id$).

Definition 1.2 *An affine structure on $\mathfrak{g}$ is called complete if the endomorphism*

$$
\begin{array}{cccc}
\theta_X : & A(\mathfrak{g}) & \to & A(\mathfrak{g}) \\
& Y & \mapsto & Y + Y.X
\end{array}
$$

is bijective for every $X \in A(\mathfrak{g})$.

This is equivalent to one of the following properties .

a)
$$R_X : \quad \begin{array}{ccc} A(\mathfrak{g}) & \to & A(\mathfrak{g}) \\ Y & \mapsto & Y.X \end{array}$$

is nilpotent for all $X \in A(\mathfrak{g})$

b) $tr(R_X) = 0$ for all $X \in A(\mathfrak{g})$.

Remarks

1. In [13] we have given examples of non complete affine structures on some nilpotent Lie algebras of maximal class. In particular, on the 3-dimensional Heisenberg algebra the affine structure associated to the following representation

$$\begin{pmatrix} a(x_1 + x_2) & a(x_1 + x_2) & 0 & x_1 \\ a(x_1 + x_2) & a(x_1 + x_2) & 0 & x_2 \\ \alpha x_1 + (\beta - 1)x_2 & \beta x_1 + (\alpha + 1)x_2 & 0 & x_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is non complete.

2. The complete affine structure on $\mathfrak{g}$ corresponds to the simply-transitive affine action of the connected corresponding Lie group G.

2   Affine structure on abelian Lie algebras

1. Let $\mathfrak{g}$ be a real abelian Lie algebra. If $\mathfrak{g}$ is provided with an affine structure then the left symmetric algebra $A(\mathfrak{g})$ is a commutative and associative real algebra. In fact we have

$$X.Y - Y.X = 0 = [X, Y]$$

and

$$X.(Y.Z) - Y.(X.Z) = ([X, Y].Z) = 0.$$

This gives

$$X.(Z.Y) = (X.Z).Y$$

and $A(\mathfrak{g})$ is associative.

Let Ψ_1 and Ψ_2 be two affine structures on $\mathfrak{g}$. They are affinely equivalent if and only if the corresponding commutative and associative algebras are isomorphic. Thus the classification of affine structures on abelian Lie algebras corresponds to the classification of commutative and associative (unitary or not) algebras. If the affine structure is complete, the endomorphisms R_X are nilpotent. As $A(\mathfrak{g})$ is also commutative, it is a nilpotent associative commutative algebra. The classification of complete affine structures on abelian Lie algebras corresponds to the classification of nilpotent associative algebras. In this frame, we can cite the works of Gabriel [7] and Mazzola [10] who study the varieties of unitary associative complex laws and give their classification for dimensions less than 5, as well as the works of Makhlouf and Cibils who study the deformations of these laws and propose classifications for nilpotent associative complex algebras [11].

3 Rigid affine structures

3.1 Definition

Let us consider a fixed basis $\{e_1, ..., e_n\}$ of the vector space $\mathbb{R}^n$. An associative law on $\mathbb{R}^n$ is given by a bilinear mapping

$$\mu : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

satisfying $\mu(e_i, \mu(e_j, e_k)) = \mu(\mu(e_i, e_j), e_k)$. If we put $\mu(e_i, e_j) = \sum C_{ij}^k e_k$, then the structural constants C_{ij}^k satisfy

$$(1) \quad \sum_l C_{il}^s C_{jk}^l - C_{ij}^l C_{lk}^s = 0, \qquad s = 1, ..., n.$$

Moreover if μ is commutative, we have

$$(2) \quad C_{ij}^k = C_{ji}^k$$

Thus the set of associative laws on $\mathbb{R}^n$ is identified to the real algebraic set embedded in $\mathbb{R}^{n^3}$, defined by the polynomial equations (1) and (2). We note this set $\mathcal{A}^c(n)$.

The law μ is unitary if there exists an $e \in \mathbb{R}^n$ such that $\mu(e, x) = x$. The set of unitary laws of $\mathcal{A}^c(n)$ is noted by $\mathcal{A}_1^c(n)$.

The linear group $GL(n, \mathbb{R})$ acts on $\mathcal{A}^c(n)$:

$$\begin{array}{ccc} GL(n, \mathbb{R}) \times \mathcal{A}^c(n) & \to & \mathcal{A}^c(n) \\ (f, \mu) & \mapsto & \mu_f \end{array}$$

where $\mu_f(e_i, e_j) = f^{-1}\mu(f(e_i), f(e_j))$.

We note by $\theta(\mu)$ the orbit of μ by this action. The orbit is isomorphic to the homogeneous space $\frac{GL(n, \mathbb{R})}{G_\mu}$, where $G_\mu = \{f \in GL(n, \mathbb{R}) \ / \ \mu_f = \mu\}$. The topology of $\mathcal{A}^c(n)$ is the induced topology of $\mathbb{R}^{n^3}$.

Definition 3.1 *The law $\mu \in \mathcal{A}^c(n)$ is rigid if $\theta(\mu)$ is open in $\mathcal{A}^c(n)$.*

Let μ be a real associative algebra and let us note by $\mu_\mathbb{C}$ the corresponding complex associative algebra. If μ is rigid in $\mathcal{A}^c(n)$ then either $\mu_\mathbb{C}$ is rigid in the scheme Ass_n of complex associative law, or $\mu_\mathbb{C}$ admits a deformation $\widetilde{\mu}_\mathbb{C}$ which is never the complexification of a real associative algebra.

This topological approach of the variety of associative algebras allows to introduce the notion of rigidity on the affine structures.

Definition 3.2 *An affine structure Ψ on an abelian Lie algebra $\mathfrak{g}$ is called rigid if the corresponding associative algebra $A(\mathfrak{g})$ is rigid in $\mathcal{A}^c(n)$.*

3.2 Cohomological approach

It is well known that a sufficient condition for an associative algebra $\mathfrak{a}$ to be rigid in $\mathcal{A}(n)$ is $H^2(\mathfrak{a}, \mathfrak{a}) = 0$, where $H^*(\mathfrak{a}, \mathfrak{a})$ is the Hochschild cohomology of $\mathfrak{a}$. Suppose that $\mathfrak{a}$ is commutative. If μ is the bilinear mapping defining the product of $\mathfrak{a}$, then $\mathfrak{a}$ is rigid if every deformation $\mu_t = \mu + \sum t^i \varphi_i$ of μ is isomorphic to μ. By the commutativity of μ we can assume that every φ_i is a symmetric bilinear mapping. Then considering only the symmetric cochains φ, we can define the commutative Harrison cohomology of μ as follows :

$$Z_s^2(\mu, \mu) = \{\varphi \in Sym(\mathbb{R}^3 \times \mathbb{R}^3, \mathbb{R}^3) \quad | \quad \delta_\mu \varphi = 0\}$$

where

$$\delta_\mu \varphi(x, y, z) = \mu(x, \varphi(y, z)) - \varphi(\mu(x, y), z) - \mu(\varphi(x, y), z) + \varphi(x, \mu(y, z)).$$

We can note that for every $f \in End(\mathbb{R}^3)$ the coboundary $\delta_\mu f \in Sym(\mathbb{R}^3 \times \mathbb{R}^3, \mathbb{R}^3)$ and then $\delta_\mu f \in Z_s^2(\mu, \mu)$.

Proposition 3.1 *If $H_s^2(\mu, \mu) = \frac{Z_s^2(\mu, \mu)}{B_s^2(\mu, \mu)} = \{0\}$, the corresponding affine structure on $\mathbb{R}^3$ is rigid.*

4 Affine structures on the 2-dimensional abelian Lie algebra

4.1 Classification of commutative associative 2-dimensional real algebras

• Suppose that the algebra A is unitary. Then its law is isomorphic to

$$\begin{cases} \mu_1(e_1, e_1) = e_1 \\ \mu_1(e_1, e_2) = e_2 \\ \mu_1(e_2, e_1) = e_2 \\ \mu_1(e_2, e_2) = e_2 \end{cases} \quad ; \quad \begin{cases} \mu_2(e_1, e_1) = e_1 \\ \mu_2(e_1, e_2) = e_2 \\ \mu_2(e_2, e_1) = e_2 \\ \mu_2(e_2, e_2) = 0 \end{cases} \quad ; \quad \begin{cases} \mu_3(e_1, e_1) = e_1 \\ \mu_3(e_1, e_2) = e_2 \\ \mu_3(e_2, e_1) = e_2 \\ \mu_3(e_2, e_2) = -e_1 \end{cases}$$

• Suppose that A is nilpotent (and not unitary). The law is isomorphic to

$$\mu_4(e_1, e_1) = e_2 \quad ; \quad \mu_5 = 0$$

• Suppose that A is non nilpotent and non unitary. Then its law is isomorphic to

$$\mu_6(e_1, e_1) = e_1.$$

4.2 Description of the affine structures

Proposition 4.1 *There are 6 affinely non-equivalent affine structures on the 2-dimensional abelian Lie algebra.*

In the following table we give the affine structures on the 2-dimensional Lie algebra, the corresponding action and precise the completeness or not of these structures.

		affine structure	affine action	complete
A_1		$\begin{pmatrix} a & 0 & a \\ b & a+b & b \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} e^a & 0 & e^a-1 \\ e^a(e^b-1) & e^ae^b & e^a(e^b-1) \\ 0 & 0 & 1 \end{pmatrix}$	NO
A_2		$\begin{pmatrix} a & 0 & a \\ b & a & b \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} e^a & 0 & e^a-1 \\ be^a & e^a & be^a \\ 0 & 0 & 1 \end{pmatrix}$	NO
A_3		$\begin{pmatrix} a & -b & a \\ b & a & b \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} e^a\cos b & -e^a\sin b & 1-e^a\cos b \\ e^a\sin b & e^a\cos b & e^a\sin b \\ 0 & 0 & 1 \end{pmatrix}$	NO
A_4		$\begin{pmatrix} 0 & 0 & a \\ a & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & a \\ a & 1 & \frac{a^2}{2}+b \\ 0 & 0 & 1 \end{pmatrix}$	YES
A_5		$\begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$	YES
A_6		$\begin{pmatrix} a & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} e^a & 0 & e^a-1 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$	NO

4.3   On the group of affine transformations

To each affine structure A_i corresponds a flat affine connection without torsion ∇^i on the abelian Lie group G.

The set of all affine transformations of (G, ∇^i) is a Lie group, noted $Aff(G, \nabla^i)$. Its Lie algebra is the set of complete affine vector fields ([3]), that is, complete vector field satisfying

$$\left[X, \nabla^i_Y Z\right] = \nabla^i_{[X,Y]}Z + \nabla^i_Y\left[X, Z\right]$$

If ∇^i is complete (in our case $i = 4, 5$) then the Lie algebra of $Aff(G, \nabla^i)$ is the Lie algebra of affine vector fields $aff(G, \nabla^i)$ and $Aff(G, \nabla^i)$ acts transitively on G.

Let us consider the corresponding affine action to ∇_i described in the previous section. The translation part defines an open set $U_i \subset \mathbb{R}^2$. For the complete case we obtain $U_i = \mathbb{R}^2$. For the non complete case we have

$$\begin{aligned} U_1 &= U_2 = U_6 = \left\{(x,y) \in \mathbb{R}^2 \,/\, x > -1\right\} \\ U_3 &= \left\{(x,y) \in \mathbb{R}^2 ,\, (x,y) \neq (1,0)\right\} \end{aligned}$$

If ϕ is an affine transformation which leaves U_i invariant, for $i = 1, 2, 6$ the matrix of ϕ is seen to have the following form

$$\begin{pmatrix} a & 0 & e^t - 1 \\ b & c & u \\ 0 & 0 & 1 \end{pmatrix}.$$

The group $Aff(G, \nabla^i)$ is the maximal group included in the semi group generated by the previous matrices.

Then the group $Aff(G, \nabla^i) = B_2 \times \mathbb{R}^2$, where B_2 is the subgroup of $GL(2, \mathbb{R})$ constituted of triangular matrices.

4.4   Rigid affine structures on the 2-dimensional abelian Lie algebra

Lemma 4.2 *Every infinitesimal deformation of an unitary associative algebra in $\mathcal{A}^c(n)$ is unitary.*

The proof is based on the study of perturbations of idempotent elements made in [11]. Let X be an idempotent element in an associative algebra of law μ. The operators

$$l_X : Y \longrightarrow XY$$

and

$$r_X : Y \longrightarrow YX$$

are simultaneously diagonalizable and the eigenvalues are respectively (1,...,1,0, ...,0) and (1,1,1,...,1,0,...,0). This set of eigenvalues is called bisystem associated to X. If X corresponds to the identity, the bisystem is $\{(1, ..., 1) (1, ..., 1)\}$.

From [11], if μ' is a perturbation of μ, there exists in μ' an idempotent element X' such that $b(X') = b(X)$. Consider the perturbation of the identity in μ'. As the bisystem corresponds to $\{\{1, ..., 1\}, \{1, ..., 1\}\}$, we can conclude that X' is the identity of μ'.

Consequence. The set of unitary associative algebra is open in $\mathcal{A}^c(n)$.

Let us note by $\theta(\mu)$ the orbit of the law μ in $\mathcal{A}^c(n)$. From the previous classification, we see that

$$\mu_i \in \overline{\theta(\mu_1)}$$

for $i = 2, 5, 6$. Then we have

Theorem 4.3 *The affine structures A_1 and A_3 on the 2-dimensional abelian Lie algebra are rigid. The other structures can be deformed into A_1 or A_3.*

Consequence. Not any complete affine structure is rigid.

4.5 Invariant affine structure on $\mathbb{T}^2$

The compact abelian Lie group $\mathbb{T}^2$ is defined by $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ identifying (x, y) with $(x + p, y + q)$, $(p, q) \in \mathbb{Z}^2$.

Proposition 4.4 *Only the structures A_4 and A_5 induce affine structures on the torus $\mathbb{T}^2$.*

Proof. It is easy to see that the affine action associated to A_1, A_2, A_3 and A_6 are incompatible with the lattice defined by $\mathbb{Z}^2$. Thus only complete structures provide affine structures on $\mathbb{T}^2$. For A_4 we obtain the following affine transformations

$$(x, y) \rightarrow \left(x + p, px + y + \left(q + p^2\right)\right)$$

This structure on $\mathbb{T}^2$ is not euclidean. For A_5 the affine structure on $\mathbb{T}^2$ which is euclidean corresponds to the transformations

$$(x, y) \rightarrow (x + p, y + q)$$

Remark 4.1 *In this proposition we find again, for the particular case of the torus, a classical result of Kuiper [9] giving the classification of affine structures on surfaces.*

5 Affine structures on the 3-dimensional abelian Lie algebra

5.1 Classification of 3-dimensional commutative associative real algebras

Let us begin by describing the classification of real associative commutative algebras. Let $\mathfrak{a}$ be a 3-dimensional (not necessarily unitary) real commutative associative algebra.

If $\mathfrak{a}$ is simple, then $\mathfrak{a}$ is, following Wedderburn's theorem, isomorphic to $(M_1(\mathbb{R}))^3$, $M_1(\mathbb{R}) \oplus M_1(\mathbb{C})$, where $M_n(D)$ is a matrix algebra on a division algebra on $\mathbb{R}$, that is $D = \mathbb{R}$ or $\mathbb{C}$. This gives the following algebras

$$\begin{cases} \mu_1(e_1, e_i) = e_i & i = 1, 2, 3 \\ \mu_1(e_i, e_1) = e_i & i = 1, 2, 3 \\ \mu_1(e_2, e_2) = e_2 \\ \mu_1(e_3, e_3) = e_3 \end{cases} \quad , \quad \begin{cases} \mu_2(e_1, e_i) = e_i & i = 1, 2, 3 \\ \mu_2(e_i, e_1) = e_i & i = 1, 2, 3 \\ \mu_2(e_2, e_2) = e_2 \\ \mu_2(e_3, e_3) = e_2 - e_1 \end{cases}$$

If $\mathfrak{a}$ is not simple, then $\mathfrak{a} = J(\mathfrak{a}) \oplus \mathfrak{s}$, where $\mathfrak{s}$ is simple and $J(\mathfrak{a})$ is the Jacobson radical of $\mathfrak{a}$. If $\mathfrak{s} = (M_1(\mathbb{R}))^2$, we obtain

$$\begin{cases} \mu_3(e_1, e_i) = e_i & i = 1, 2, 3 \\ \mu_3(e_i, e_1) = e_i & i = 1, 2, 3 \\ \mu_3(e_2, e_2) = e_2 \end{cases}$$

If $\mathfrak{s} = M_1(\mathbb{C})$

$$\begin{cases} \mu_4(e_1, e_i) = e_i & i = 1, 2, 3 \\ \mu_4(e_i, e_1) = e_i & i = 1, 2, 3 \\ \mu_4(e_3, e_3) = e_2 \end{cases}$$

where $J(\mathfrak{a})$ is not abelian or

$$\begin{cases} \mu_5(e_1, e_i) = e_i & i = 1, 2, 3 \\ \mu_5(e_i, e_1) = e_i & i = 1, 2, 3 \end{cases}$$

where $J(\mathfrak{a})$ is abelian.

Suppose that $\mathfrak{a}$ is not unitary. As the Levi decomposition also holds in this case, we have the following possibilities : $\mathfrak{a} \simeq (M_1(\mathbb{R}))^2 \oplus J(\mathfrak{a})$ or $M_1(\mathbb{C}) \oplus J(\mathfrak{a})$. This gives the following algebras

$$\begin{cases} \mu_6(e_1, e_1) = e_1 \\ \mu_6(e_2, e_2) = e_2 \end{cases} \qquad , \qquad \begin{cases} \mu_7(e_1, e_1) = e_1 \\ \mu_7(e_1, e_2) = e_2 \\ \mu_7(e_2, e_1) = e_2 \\ \mu_7(e_2, e_2) = -e_1 \end{cases}$$

$$\{\mu_8(e_1, e_1) = e_1 \qquad , \qquad \begin{cases} \mu_9(e_1, e_1) = e_1 \\ \mu_9(e_1, e_2) = e_2 \\ \mu_9(e_2, e_1) = e_2 \end{cases}$$

$$\begin{cases} \mu_{10}(e_1, e_1) = e_1 \\ \mu_{10}(e_2, e_2) = e_3 \end{cases}$$

If moreover $\mathfrak{a}$ is nilpotent, then it is isomorphic to one of the following algebras

$$\begin{cases} \mu_{11}(e_1, e_1) = e_2 \\ \mu_{11}(e_3, e_3) = e_2 \end{cases} \qquad , \qquad \begin{cases} \mu_{12}(e_1, e_1) = e_2 \\ \mu_{12}(e_3, e_3) = -e_2 \end{cases}$$

$$\{\mu_{13}(e_1, e_1) = e_2 \qquad , \qquad \begin{cases} \mu_{14}(e_1, e_1) = e_2 \\ \mu_{14}(e_1, e_2) = e_3 \\ \mu_{14}(e_2, e_1) = e_3 \end{cases}$$

$$\{\mu_{15}(e_i, e_j) = 0 \qquad i, j \in \{1, 2, 3\}$$

Theorem 5.1 *Every 3-dimensional real commutative associative Lie algebra $\mathfrak{a}$ is isomorphic to one of the algebras $\mathfrak{a}_i$, $i = 1, 2, ..., 15$.*
If $\mathfrak{a}$ is nilpotent, $\mathfrak{a}$ is isomorphic to $\mathfrak{a}_i$, $i = 11, ..., 15$.

5.2   Affine structures on $\mathbb{R}^3$

For each associative algebra of the previous classification we can describe the affine action on $\mathbb{R}^3$.

Theorem 5.2 *There exist 15 invariant affinely non-equivalent affine structures on the 3-dimensional abelian Lie algebra. They are given by :*

A_1	$(x, y, z) \rightarrow$	$\begin{cases} e^a x + e^a - 1, \\ e^a \left(e^b - 1\right) x + e^{a+b} y + e^a (e^b - 1), \\ e^a \left(e^c - 1\right) x + e^{a+c} z + e^a \left(e^c - 1\right) \end{cases}$
A_2	$(x, y, z) \rightarrow$	$\begin{cases} x e^a \cos c - z e^a \sin c + e^a \cos c - 1, \\ \left(-e^a \cos c + e^{a+b}\right) x + e^{a+b} y + z e^a \sin c - e^a \cos c + e^{a+b}, \\ x e^a \sin c + z e^a \cos c + e^a \sin c \end{cases}$
A_3	$(x, y, z) \rightarrow$	$\begin{cases} e^a x + e^a - 1, \\ e^a \left(e^b - 1\right) x + e^{a+b} y + e^a \left(e^b - 1\right), \\ c e^a x + e^a z + c e^a \end{cases}$
A_4	$(x, y, z) \rightarrow$	$\begin{cases} e^a x - 1 + e^a, \\ (b + \frac{c^2}{2}) e^a x + e^a y + c e^a z + (b + \frac{c^2}{2}) e^a, \\ c e^a x + e^a z + c e^a \end{cases}$
A_5	$(x, y, z) \rightarrow$	$\begin{cases} e^a x - 1 + e^a, \\ b e^a x + e^a y + b e^a, \\ c e^a x + e^a z + c e^a \end{cases}$
A_6	$(x, y, z) \rightarrow$	$\begin{cases} e^a x - 1 + e^a, \\ e^b y - 1 + e^b, \\ z + c \end{cases}$
A_7	$(x, y, z) \rightarrow$	$\begin{cases} x e^a \cos b - y e^a \sin b - 1 + e^a \cos b, \\ x e^a \sin b + y e^a \cos b + e^a \sin b, \\ z + c \end{cases}$
A_8	$(x, y, z) \rightarrow$	$\begin{cases} e^a x - 1 + e^a, \\ y + b, \\ z + c \end{cases}$
A_9	$(x, y, z) \rightarrow$	$\begin{cases} e^a x - 1 + e^a, \\ b e^a x + e^a y + b e^a, \\ z + c \end{cases}$
A_{10}	$(x, y, z) \rightarrow$	$\begin{cases} e^a x - 1 + e^a, \\ y + b, \\ b y + z + \frac{b^2}{2} c \end{cases}$
A_{11}	$(x, y, z) \rightarrow$	$\begin{cases} x + a, \\ a x + y + c z + b + \frac{1}{2}(a^2 + c^2), \\ z + c \end{cases}$
A_{12}	$(x, y, z) \rightarrow$	$\begin{cases} x + a, \\ a x + y - c z + b + \frac{1}{2}(a^2 - c^2), \\ z + c \end{cases}$

A_{13}	$(x,y,z) \rightarrow \begin{cases} x+a, \\ ax+y+b+\frac{a^2}{2}, \\ z+c \end{cases}$
A_{14}	$(x,y,z) \rightarrow \begin{cases} x+a, \\ ax+y+b+\frac{a^2}{2}, \\ (b+\frac{a^2}{2})x+ay+z+\frac{a^3}{6}+ab+c \end{cases}$
A_{15}	$(x,y,z) \rightarrow \begin{cases} x+a, \\ y+b, \\ z+c \end{cases}$

where $a,b,c \in \mathbb{R}$. Only the structures A_i, $i = 11,..,15$ are complete.

5.3   Rigid affine structures

Recall that an affine structure on $\mathbb{R}^3$ is called rigid if the corresponding commutative and associative real algebra is rigid in the algebraic variety $\mathcal{A}^c(3)$.

As the laws μ_1 and μ_2 are semi-simple associative, their second cohomological group is trivial. These structures are rigid.

Consider the remaining laws μ_i. We can easily compute the linear space $H_s^2(\mu,\mu)$ and presente the results in the following tables :

Unitary case :

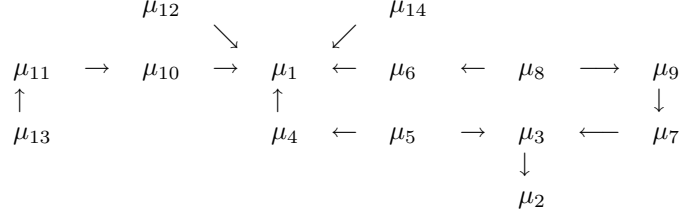| laws	$dimH_s^2(\mu,\mu)$	basis of $H_s^2(\mu,\mu)$
μ_3	1	$\varphi(e_3,e_3) = e_2 - e_1$
μ_4	2	$\begin{cases} \varphi_1(e_2,e_2) = e_2 \\ \varphi_2(e_2,e_3) = e_{3.} \end{cases}$
μ_5	4	$\begin{cases} \varphi_1(e_2,e_2) = e_2 \\ \varphi_2(e_2,e_3) = e_{3.} \end{cases} ; \begin{cases} \varphi_3(e_3,e_3) = e_2 \\ \varphi_4(e_3,e_3) = e_{3.} \end{cases}$

Non unitary case :

laws	$dimH_s^2$	basis of $H_s^2(\mu,\mu)$
μ_6	1	$\varphi_1(e_3,e_3)=e_3$
μ_7	1	$\varphi_1(e_3,e_3)=e_3$
μ_8	6	$\left\{\begin{array}{l}\varphi_1(e_2,e_2)=e_2\\\varphi_2(e_2,e_2)=e_3.\end{array}\right.$; $\left\{\begin{array}{l}\varphi_3(e_3,e_3)=e_2\\\varphi_4(e_3,e_3)=e_3.\end{array}\right.$; $\left\{\begin{array}{l}\varphi_5(e_2,e_3)=e_2\\\varphi_6(e_3,e_3)=e_3.\end{array}\right.$
μ_9	3	$\left\{\begin{array}{l}\varphi_1(e_2,e_2)=e_1\\\varphi_2(e_2,e_3)=e_2.\end{array}\right.$; $\varphi_3(e_3,e_3)=e_3$
μ_{10}	1	$\varphi_1(e_2,e_3)=e_2,\varphi_1(e_3,e_3)=e_3.$

Nilpotent and complete case :

laws	$dimH_s^2$	basis of $H_s^2(\mu,\mu)$
μ_{11}	3	$\left\{\begin{array}{l}\varphi_1(e_1,e_3)=e_1\\\varphi_1(e_2,e_3)=e_2.\end{array}\right.$; $\left\{\begin{array}{l}\varphi_2(e_1,e_3)=e_3\\\varphi_2(e_3,e_3)=e_1.\end{array}\right.$; $\left\{\varphi_3(e_3,e_3)=e_3.\right.$
μ_{12}	4	$\begin{array}{l}\varphi_1(e_1,e_2)=e_2.\\\varphi_2(e_3,e_3)=e_3.\end{array}$; $\left\{\begin{array}{l}\varphi_3(e_1,e_3)=e_3\\\varphi_3(e_3,e_3)=e_1.\end{array}\right.$; $\left\{\begin{array}{l}\varphi_4(e_1,e_1)=e_3\\\varphi_4(e_1,e_3)=e_1\\\varphi_4(e_2,e_3)=2e_2.\end{array}\right.$
μ_{13}	7	$\left\{\begin{array}{l}\varphi_1(e_1,e_2)=e_1\\\varphi_1(e_2,e_2)=e_2.\\\varphi_2(e_1,e_2)=e_2.\end{array}\right.$; $\left\{\begin{array}{l}\varphi_3(e_1,e_2)=e_3\\\varphi_4(e_1,e_3)=e_1\\\varphi_4(e_2,e_3)=e_2.\end{array}\right.$ $\begin{array}{l}\varphi_5(e_1,e_3)=e_3.\\\varphi_6(e_3,e_3)=e_2.\\\varphi_7(e_3,e_3)=e_3.\end{array}$
μ_{14}	3	$\left\{\begin{array}{l}\varphi_1(e_1,e_3)=e_1\\\varphi_1(e_2,e_2)=e_1\\\varphi_1(e_2,e_3)=e_2\\\varphi_1(e_3,e_3)=e_3.\end{array}\right.$; $\left\{\begin{array}{l}\varphi_2(e_1,e_3)=e_2\\\varphi_2(e_2,e_2)=e_2\\\varphi_2(e_2,e_3)=e_3.\end{array}\right.$; $\left\{\begin{array}{l}\varphi_3(e_1,e_3)=e_3\\\varphi_3(e_2,e_2)=e_3.\end{array}\right.$
μ_{15}	18	

In this box we suppose that $\varphi(e_i, e_j) = \varphi(e_j, e_i)$ and the non defined $\varphi(e_s, e_t)$ are equal to zero.

We will note by $\mu_i \rightarrow \mu_j$ when $\mu_i \in \overline{\mathcal{O}(\mu_j)}$. We obtain the following diagramm

$$
\begin{array}{ccccccccccc}
& & \mu_{12} & & & & \mu_{14} & & & & \\
& & & \searrow & & \swarrow & & & & & \\
\mu_{11} & \rightarrow & \mu_{10} & \rightarrow & \mu_1 & \leftarrow & \mu_6 & \leftarrow & \mu_8 & \longrightarrow & \mu_9 \\
\uparrow & & & & \uparrow & & & & & & \downarrow \\
\mu_{13} & & & & \mu_4 & \leftarrow & \mu_5 & \rightarrow & \mu_3 & \longleftarrow & \mu_7 \\
& & & & & & & & \downarrow & & \\
& & & & & & & & \mu_2 & &
\end{array}
$$

Theorem 5.3 *There exist two rigid affine structures on the 3-dimensional abelian Lie algebra. They are the structures associated to the semi simple associative algebras μ_1 and μ_2.*

5.4   Invariant affine structures on $\mathbb{T}^3$

We can see that the affine actions A_1 to A_{10} are compatible with the action of $\mathbb{Z}^3$ on $\mathbb{R}^3$ if the exponentials which appear in the analytic expressions of the affine transformations are equal to 1. This gives the identity for A_1 and A_3. As A_2 is incompatible with the action of $\mathbb{Z}^3$ for any values of the parameters a, b, c, the affine structures corresponding to the unitary cases are given by A_4 and A_5 for $a = 0$. This corresponds to the following affine structures on the torus $\mathbb{T}^3$:

$$
(x, y, z) \in \mathbb{R}^3/\mathbb{Z}^3 \rightarrow \left\{
\begin{array}{l}
x \\
px + y + qz + p \\
qx + z + q
\end{array}
\right.
$$

and

$$
(x, y, z) \in \mathbb{R}^3/\mathbb{Z}^3 \rightarrow \left\{
\begin{array}{l}
x \\
px + y + p \\
qx + z + q
\end{array}
\right.
$$

with $p, q \in \mathbb{Z}$.

For the actions A_6 to A_{10}, they induce affine actions on the torus if $a = b = 0$ for A_6 and $a = 0$ for A_7 to A_{10} but this appears as a particular case of A_{11}, A_{13} and A_{14}. Let us examine the complete and nilpotent cases; we find the following

affine structure on $\mathbb{T}^3$:

$(x,y,z) \in \mathbb{R}^3/\mathbb{Z}^3 \rightarrow$	$\begin{cases} x+p, \\ px+y+rz+q, \\ z+r \end{cases}$
$(x,y,z) \in \mathbb{R}^3/\mathbb{Z}^3 \rightarrow$	$\begin{cases} x+p, \\ px+y-rz+q, \\ z+r \end{cases}$
$(x,y,z) \in \mathbb{R}^3/\mathbb{Z}^3 \rightarrow$	$\begin{cases} x+p, \\ px+y+q, \\ z+r \end{cases}$
$(x,y,z) \in \mathbb{R}^3/\mathbb{Z}^3 \rightarrow$	$\begin{cases} x+p, \\ px+y+q, \\ qx+py+z+r \end{cases}$
$(x,y,z) \in \mathbb{R}^3/\mathbb{Z}^3 \rightarrow$	$\begin{cases} x+p, \\ y+q, \\ z+r \end{cases}$

Theorem 5.4 *There exist 7 affine structures on the torus $\mathbb{T}^3$. They correspond to the following affine crystallographic subgroups of $Aff(\mathbb{R}^3)$:*

$\Gamma_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ p & 1 & q & p \\ q & 0 & 1 & q \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$	$\Gamma_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ p & 1 & 0 & p \\ q & 0 & 1 & q \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$
$\Gamma_3 = \left\{ \begin{pmatrix} 1 & 0 & 0 & p \\ p & 1 & r & q \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$	$\Gamma_4 = \left\{ \begin{pmatrix} 1 & 0 & 0 & p \\ p & 1 & -r & q \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$
$\Gamma_5 = \left\{ \begin{pmatrix} 1 & 0 & 0 & p \\ p & 1 & 0 & q \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$	$\Gamma_6 = \left\{ \begin{pmatrix} 1 & 0 & 0 & p \\ p & 1 & 0 & q \\ q & p & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$
$\Gamma_7 = \left\{ \begin{pmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$	

Remark 5.1 *The (complete) nilpotent and unimodular cases are completely classified in [6].*

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